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Bruhat canonical form for linear systems

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Abstract

Based on a generalization of the classical Bruhat factorization of nonsingular matrices to arbitrary rectangular matrices, a new canonical form for state space equivalence of controllable and observable linear systems is introduced. The proposed canonical form is shown to be closely related to a canonical form due to Bosgra and van der Weiden. Moreover, in the single-input single-output case and up to minor details, the proposed normal form is equivalent to the continued fraction canonical form introduced by Kalman. Connections to a cell decomposition by Fuhrmann and Krishnaprasad are discussed as well. Discrete invariants appearing in the Bruhat canonical form are shown to be invariants for restricted output feedback equivalence.

Keywords: Linear systems; Bruhat decomposition; Block Hankel matrices; Canonical forms; Cell decompositions

1. Introduction

The analysis of canonical forms and parameterization problems has a long tradition in linear systems theory, with interesting applications to e.g. linear algebra [14], topology [24,29] and moduli spaces of rational curves [16]. In this paper we take up an old theme from linear systems theory, i.e. the construction of state space canonical forms. Thus we consider the space $\mathrm{L}_n^{p,m}$ of controllable and observable linear systems

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$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t)$$

modulo state space similarity. Here, A, B, C denote, respectively, $n \times n$, $n \times m$, $p \times n$ matrices with real entries. The space $\mathrm{L}_n^{p,m}$ is of central importance in linear systems theory, since the quotient space of $\mathrm{L}_n^{p,m}$ relative to the equivalence relation of *similarity*

$$(A, B, C) \mapsto (SAS^{-1}, SB, CS^{-1})$$

can be identified with the space $\mathrm{Rat}(n, m, p)$ of $p \times m$ strictly proper rational transfer matrices

$$G(s) = C(sI - A)^{-1}B$$

of McMillan degree n. Equivalently, by the main theorem of realization theory [22,26], there is a one to one correspondence between the quotient space of $\mathrm{L}_n^{p,m}$ (relative to similarity) and the space $\mathrm{Hank}_{MN}(n, m, p)$ of $M \times N$ block Hankel matrices $H = [H_{i+j-1}]_{i,j=1}^{M,N}$, $H_{i+j-1} \in \mathbb{R}^{p\times m}$, satisfying

$$\operatorname{rank} H = \operatorname{rank}[H_{i+j-1}]_{i,j=1}^{n} = n,$$

provided that $M, N \geqslant n$ and $\max\{M, N\} > n$. Thus, from a geometric viewpoint, we are interested in parameterizing spaces of finite rank block Hankel matrices or, alternatively, of degree n rational curves in a Grassmann manifold, satisfying a base point condition [14]. For first contributions to the geometry of rational transfer functions we refer to [5]. However, the real starting point for this work has been the contribution by Fuhrmann and Krishnaprasad [11], where for $m = p = 1$, a decomposition of $\mathrm{Rat}(n, 1, 1)$ into cells was proposed using continued fraction expansions. In subsequent work [18,15,19] it has been shown that the Fuhrmann–Krishnaprasad cell construction defines in fact a cell decomposition in the usual topological sense.

The origins of state space canonical forms on $\mathrm{L}_n^{p,m}$ go back to the 1970s and early 1980s [25]. The celebrated Fuhrmann realization [8,9] associates with every factorization $G(s) = V(s)T^{-1}(s)U(s)$ of a strictly proper transfer function $G(s)$ a canonical state space realization (A, B, C), using the shift operator on polynomial model spaces. Moreover, controllability and observability properties of (A, B, C) are shown to be equivalent to left and right coprimeness properties of (T, U) and (V, T), respectively. Despite this extremely elegant approach to state space realization, the controllability and observability properties of such canonical realizations are not easily reflected in the normal form structure of (A, B, C). This is in contrast to the well-known controller or observer canonical forms [21], where the system matrices (A, B, C) have just 0, 1 or completely free parameters $*$ as entries and controllability or observability is obvious by construction. For controllable and observable realizations (A, B, C) such simple 0, 1, $*$-canonical forms were not known for a long time. In the single-input single-output case, Kalman [23] has been the first to construct a canonical form for state space similarity of controllable and observable systems. His form involved 0, 1-entries as well as arbitrary parameters $*$, some of them being constrained to be nonzero. Fuhrmann and Krishnaprasad [11] put this work into a geometric perspective using cell decompositions. Subsequently, Antoulas and Bishop extended this work to the multivariable case using matrix continued fraction expansions; see [1,10]. The above work was mainly done using rational functions and associated coprime factorizations. Factorizations of Hankel matrices were first used by Bosgra and van der Weiden in their 1980 paper [3] to define a canonical form for controllable and observable systems; see also [12]. The Bosgra–van der Weiden form involves three discrete invariants for system equivalence: the controllability and observability indices as well a canonical permutation matrix. By fixing these invariants, a canonical form is constructed in which only entries 0, 1 or free parameters $*$ appear (that might additionally be restricted to be nonzero). By construction, the realizations are automatically controllable and observable. Moreover, the structure of the canonical form is more explicit than compared with [1]. These are considerable advantages compared to the previously known canonical forms and offer e.g. new approaches to the analysis of partial realizations [2]. A disadvantage though is that the construction of the canonical form by Bosgra and van der Weiden is rather unintuitive, using complicated matrix manipulations.

In this paper we offer a new approach to the construction of the Bosgra–van der Weiden form, using the Bruhat decomposition of finite block Hankel matrices as a starting point. This approach turns out to be very fruitful. It allows us to obtain a canonical form on $\mathrm{L}_n^{p,m}$ relative to similarity which has not been investigated previously (Sections 3 and 4). We refer to the canonical form as the *Bruhat canonical form*.

The paper is structured as follows. Section 2 extends the well-known Bruhat factorization from invertible square matrices to arbitrary rectangular matrices. In Section 3 we apply this construction to block Hankel matrices of linear systems and obtain a special type of factorizations of Hankel matrices, that are uniquely defined by output- and input-Kronecker indices, and a permutation matrix. It is shown that these defines not only discrete invariants for state space similarity, but even for a restricted class of output feedback transformations. In Section 4 the new canonical form is constructed and canonical realizations are characterized. In Section 5 we take up the question of the relationships between the Bosgra–van der Weiden canonical form, the Bruhat canonical form and Kalman's canonical form. Its is shown, that up to one condition, the structures of the Bosgra–van der Weiden canonical form and the Bruhat canonical form are almost identical. Finally, in Section 6, we discuss some elementary topological consequences for the space of rational transfer functions.

2. Generalized Bruhat decomposition

Let $\mathrm{GL}(n)$ denote the group of nonsingular $n \times n$ matrices with real entries. Let $\mathscr{B}^+(n)$ (respectively, $\mathscr{B}^-(n)$) be the subgroup of upper (respectively, lower) triangular matrices in $\mathrm{GL}(n)$, and $\mathscr{U}^+(n)$ (respectively, $\mathscr{U}^-(n)$) the subgroup of matrices in $\mathscr{B}^+(n)$ (respectively, $\mathscr{B}^-(n)$) with unit diagonal. Denote by $\mathscr{W}(n)$ the group of $n \times n$ permutation matrices. Here an $n \times n$ matrix is a *permutation matrix* if it is obtained from the $n \times n$ identity matrix by interchanges of rows or columns.

It is a well-known result from linear algebra (see, for example, [4,7,28]) that a given matrix $A \in \mathrm{GL}(n)$ can be multiplied on the left and on the right by nonsingular lower and upper triangular matrices (corresponding to elementary row and column operations) until it becomes just a permutation matrix. With a little more care, this so-called LPU-factorization

$$A = B_1 P B_2, \quad B_1 \in \mathscr{B}^-(n), \quad P \in \mathscr{W}(n), \quad B_2 \in \mathscr{B}^+(n)$$

can be made unique. In fact, P is the unique "canonical" permutation matrix such that

$$\operatorname{rank} A_{rs} = \operatorname{rank} P_{rs} \quad \forall r, s \in \{1, \dots, n\},$$

where A_{rs} denotes the submatrix of A, consisting of the first r rows and s columns. For each $P \in \mathscr{W}(n)$ let $\mathscr{B}_P^-(n)$ denote the group

$$\mathscr{B}_P^-(n) := \mathscr{B}^-(n) \cap P\mathscr{B}^-(n)P^\top.$$

From the theory of algebraic groups the following version of the LPU-factorization is well known [17,20]. Bruhat [6] established the result 1954 in the framework of classical linear Lie groups. The *Bruhat decomposition* was then exploited in the study of finite simple groups and algebraic groups, and subsequently axiomatized by Tits [27] in his theory of *BN-pairs*.

Theorem 2.1 (Bruhat's lemma). *Every invertible matrix $A \in \mathrm{GL}(n)$ can be uniquely written as a product $A = BPU$, where $B \in \mathscr{B}_P^-(n)$, $U \in \mathscr{U}^+(n)$ and $P \in \mathscr{W}(n)$ denotes the unique canonical permutation matrix.*

In this section we extend the Bruhat decomposition of nonsingular matrices to the case of nonzero $k \times l$ matrices and discuss its extension to block Hankel matrices.

A finite set of strictly increasing positive integers is said to be an *index list*. For an index list I and a matrix A of appropriate size, A_I (respectively, A^I) is the submatrix of A formed by the rows (respectively, columns) whose indices are in I.

Definition 2.2. Let Γ be a nonempty subset of $\mathrm{GL}(n)$, and let A be an $k \times n$ (respectively, $n \times k$) matrix of full rank n. The matrix A is said to be a Γ-row (respectively, Γ-column) pattern matrix with index list $I = \{i_1, \dots, i_n\}$ if the following two conditions hold:

(i) $A_I \in \Gamma$ (respectively, $A^I \in \Gamma$);
(ii) each row (respectively, column) $i \notin I$ of A is linearly dependent on rows (respectively, columns) $1, 2, \dots, i-1$.

We illustrate this terminology by the following examples, where $\otimes$ denotes a nonzero entry of the matrix, while $\times$ represents an arbitrary unconstrained entry.

Examples 2.3. (a) Let $n = 3$, $k = 6$ and $\Gamma = \mathscr{U}^+(3)$. The matrix

$$A = \begin{bmatrix} 0 & 1 & \times & \times & \times & \times \\ 0 & 0 & 0 & 1 & \times & \times \\ 0 & 0 & 0 & 0 & 1 & \times \end{bmatrix}$$

is an $\mathscr{U}^+(3)$-column pattern matrix with index list $I = \{2, 4, 5\}$, since $A^I \in \mathscr{U}^+(3)$ and each column $i \in \{1, 3, 6\}$ is linearly dependent on columns $1, \dots, i-1$.

(b) Let $n = 3$, $k = 7$ and $\Gamma = \mathscr{B}_P^-(3)$, where $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \otimes & 0 & 0 \\ \times & 0 & 0 \\ \times & \otimes & 0 \\ \times & \times & 0 \\ \times & 0 & \otimes \\ \times & \times & \times \end{bmatrix}$$

is a $\mathscr{B}_P^-(3)$-row pattern matrix with index list $I = \{2, 4, 6\}$, since

$$A_I = \begin{bmatrix} \otimes & 0 & 0 \\ \times & \otimes & 0 \\ \times & 0 & \otimes \end{bmatrix} \in \mathscr{B}_P^-(3)$$

and each row $i \in \{1, 3, 5, 7\}$ of A is linearly dependent on rows $1, \dots, i-1$.

The next result generalizes Theorem 2.1 to the situation where A is an arbitrary nonzero matrix.

Theorem 2.4. *To any $k \times l$ matrix A of* rank $n \geqslant 1$ *there correspond index lists I and J of n integers, an $n \times n$ permutation matrix P, a $\mathscr{B}_P^-(n)$-row pattern matrix X with index list I, and an $\mathscr{U}^+(n)$-column pattern matrix Y with index list J, such that*

$$A = XPY. \tag{2.1}$$

Furthermore, I, J, P, X, Y are all determined uniquely by A.

Proof. *Existence:* Let $I = \{i_1 < \cdots < i_n\}$ and $J = \{j_1 < \cdots < j_n\}$ denote the first n linearly independent rows and columns of A, respectively. The $n \times n$ submatrix formed from A by intersection of the rows I and columns J is denoted by A_{IJ}. Define $\{i'_{n+1} < \cdots < i'_k\} := \{1, \dots, k\} \setminus I$ and $\{j'_{n+1} < \cdots < j'_l\} := \{1, \dots, l\} \setminus J$. Let Q and $\widetilde{Q}$ denote the $k \times k$ and $l \times l$ permutation matrices with 1's at the positions $(1, i_1), \dots, (n, i_n), (n+1, i'_{n+1}), \dots, (k, i'_k)$ and $(j_1, 1), \dots, (j_n, n), (j'_{n+1}, n+1), \dots, (j'_l, l)$, respectively. By the Schur complement formula,

$$QA\widetilde{Q} = \begin{bmatrix} A_{IJ} & S \\ R & RA_{IJ}^{-1}S \end{bmatrix},$$

where R, S are $(k-n) \times n$ and $n \times (l-n)$ matrices, respectively. According to Theorem 2.1, there exist unique matrices $P \in \mathscr{W}(n)$, $B \in \mathscr{B}_P^-(n)$ and $U \in \mathscr{U}^+(n)$ such that $A_{IJ} = BPU$. It follows that

$$\begin{aligned} A &= Q^{-1} \begin{bmatrix} BPU & S \\ R & RU^{-1}P^TB^{-1}S \end{bmatrix} \widetilde{Q}^{-1} \\ &= Q^{-1} \begin{bmatrix} B \\ RU^{-1}P^T \end{bmatrix} P \begin{bmatrix} U & P^TB^{-1}S \end{bmatrix} \widetilde{Q}^{-1}. \end{aligned}$$

Define

$$X := Q^{-1} \begin{bmatrix} B \\ RU^{-1}P^T \end{bmatrix}, \qquad Y := [U \quad P^TB^{-1}S]\widetilde{Q}^{-1}$$

and observe that $A = XPY$ is a full rank factorization of A. Hence, the ith row (respectively, column) of A is contained in the linear span of its preceding rows (respectively, columns) if and only if the ith row of X (respectively, column of Y) is in the linear span of its preceding rows (respectively, columns). Since $X_I = B \in \mathscr{B}_P^-(n)$ (respectively, $Y^J = U \in \mathscr{U}^+(n)$), it follows that X (respectively, Y) is a $\mathscr{B}_P^-(n)$-row (respectively, $\mathscr{U}^+(n)$-column) pattern matrix with index list I (respectively, J).

Uniqueness: Suppose we also have $A = \widetilde{X}\widetilde{P}\widetilde{Y}$, where $\widetilde{P} \in \mathscr{W}(n)$, $\widetilde{X}$ is a $\mathscr{B}_{\widetilde{P}}^-(n)$-row pattern matrix with index list $\widetilde{I}$, and $\widetilde{Y}$ is an $\mathscr{U}^+(n)$-column pattern matrix with index list $\widetilde{J}$. Since $A = XPY$ and $A = \widetilde{X}\widetilde{P}\widetilde{Y}$ are full rank factorizations of A, it follows that both I and $\widetilde{I}$ denote the first n linearly independent columns of A, forcing $I = \widetilde{I}$. Similarly it follows that $J = \widetilde{J}$. From Theorem 2.1 we conclude that $P = \widetilde{P}$, $X_I = \widetilde{X}_I$ and $Y^J = \widetilde{Y}^J$. Therefore $XPY^J = \widetilde{X}PY^J$ and $X_IPY = X_IP\widetilde{Y}$, which implies $X = \widetilde{X}$, $Y = \widetilde{Y}$. $\square$

Definition 2.5. The factorization (2.1) is called the Bruhat decomposition of A, and P is said to be the Bruhat permutation of A. The pair (I, J) of index lists is referred to as the Bruhat symbol of A.

Remark 2.6. Let A be a matrix with Bruhat symbol (I, J). The proof of Theorem 2.4 shows that I and J consist of the indices of the first n linearly independent rows and columns of A, respectively.

In [18] the Bruhat decomposition of scalar nonsingular Hankel matrices has been investigated. Here we focus on the more general task of constructing the Bruhat decomposition of arbitrary block Hankel matrices. The special situation, where the Hankel matrix is actually defined by a linear system is discussed in the next section.

Let $\mathrm{Hank}(n, Mp \times Nm)$ denote the space of $M \times N$ block Hankel matrices

$$H = \begin{bmatrix} H_1 & H_2 & \cdots & H_N \\ H_2 & & & \\ \vdots & & & \vdots \\ H_M & & \cdots & H_{M+N-1} \end{bmatrix} \tag{2.2}$$

of rank n with $p \times m$ entries H_i, $i = 1, \ldots, M + N - 1$. It follows from the special structure of Hankel matrices that only a restricted class of Bruhat symbols $(I, J) = (\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\})$, $i_n \leqslant Mp$, $j_n \leqslant Nm$, may qualify as the Bruhat symbol of a Hankel matrix $H \in \mathrm{Hank}(n, Mp \times Nm)$.

Definition 2.7. A Bruhat symbol (I, J) is called admissible, if there exists a Hankel matrix having Bruhat symbol (I, J).

In order to obtain necessary conditions for a Bruhat symbol to be admissible it is convenient to introduce the following notation and terminology. When $I = \{i_1, \ldots, i_n\}$ is an index list and p is a positive integer, the index list $I + p$ is defined by

$$I + p := \{i_1 + p, \ldots, i_n + p\}.$$

Given a block Hankel matrix H of the form (2.2), a selection of row indices of H, $I = \{i_1, \ldots, i_n\}$, $1 \leqslant i_1 < \cdots < i_n \leqslant Mp$, is said to be *nice* if and only if $i \in I$ and $i > p$ implies $i - p \in I$. An analogous terminology is used for selections of column indices of H.

Lemma 2.8. *If* $\min\{M, N\} > 1$ *and* $H \in \mathrm{Hank}(n, Mp \times Nm)$ *has Bruhat symbol* $(I, J) = (\{i_1, \ldots, i_n\}, \{j_1, \ldots, j_n\})$, *then*

(i) $i_n > (M - 1)p$ *or* J *is nice*;
(ii) $j_n > (N - 1)m$ *or* I *is nice*;
(iii) *either* $i_1 > (M - 1)p$ *and* $j_1 > (N - 1)m$ *or* $i_1 \leqslant (M - 1)p$ *and* $j_1 \leqslant (N - 1)m$.

Proof. (i) Suppose, on the contrary, that $i_n \leqslant (M - 1)p$ and J is not nice. Then there would be an index $j + m \in J$ such that $j \notin J$. Let $H = XPY$ be the Bruhat decomposition of H (see (2.1)) and write y^j for the jth column of Y. Since J indexes the first n linearly independent columns of H and $H = XPY$ is a full rank factorization, it would follow that $y^j \in \mathrm{span}\{y^k; k = 1, \ldots, j - 1\}$, so that $y^j = \sum_{k=1}^{j-1} \alpha_k y^k$ for some scalars α_k. Therefore, $X_{I+p} P y^j = \sum_{k=1}^{j-1} \alpha_k X_{I+p} P y^k$. By the Hankel structure, $X_{I+p} P y^l = X_I P y^{l+m}$ for $l = 1, 2, \ldots, (N - 1)m$, so that $X_I P y^{j+m} =$

$\sum_{k=1}^{j-1} \alpha_k X_I P y^{k+m}$. But $X_I P$ is nonsingular and so $y^{j+m} = \sum_{k=1}^{j-1} \alpha_k y^{k+m}$, which would contradict $j + m \in J$.

The proof of (ii) is similar to that of (i) and hence is omitted.

(iii) Assume that the assertion is false. Without loss of generality we could assume that $i_1 > (M-1)p$ and $j_1 \leqslant (N-1)m$. Then the rows $1, 2, \ldots, (M-1)p$ of H would be zero. By the Hankel structure, it would follow that $H_1 = H_2 = \cdots = H_{M+N-2} = 0$ contradicting $j_1 \leqslant (N-1)m$. □

The conditions (i)–(iii) of Lemma 2.8 are not sufficient for a Bruhat symbol to be admissible, as the example

$$p = 2,\ M = 3,\ m = 3,\ N = 2,\ n = 2, \qquad (I, J) = (\{3, 6\}, \{1, 4\})$$

shows. We believe that it is not easy to find conditions that characterize admissibility of Bruhat symbols for arbitrary block Hankel matrices. In the next section we show that the situation is much better if the block Hankel matrices are defined by a regular linear system.

3. Bruhat symbol and permutation for linear systems

Given a regular linear system $(A, B, C) \in \mathrm{L}_n^{p,m}$ and $M, N \geqslant 1$ we consider the associated block Hankel matrix

$$H_{MN}(A, B, C) := \begin{bmatrix} CB & CAB & \cdots & CA^{N-1}B \\ CAB & & & \\ \vdots & & & \vdots \\ CA^{M-1}B & & \cdots & CA^{M+N-2}B \end{bmatrix}.$$

The Hankel $H_{MN}(A, B, C)$ is factored as

$$H_{MN}(A, B, C) = O_M(C, A) R_N(A, B), \tag{3.1}$$

where

$$O_M := O_M(C, A) := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{M-1} \end{bmatrix}$$

and

$$R_N := R_N(A, B) := [B, \quad AB, \quad \ldots, \quad A^{N-1}B]$$

are the Mth order *observability* and Nth order *controllability matrices* of (C, A) and (A, B), respectively. Since (C, A) and (A, B) are assumed to be observable and controllable, respectively, both $O_M(C, A)$ and $R_N(A, B)$ have (full) rank n whenever $M, N \geqslant n$. Therefore their product $H_{MN}(A, B, C)$ also must have rank n whenever $M, N \geqslant n$. Hence, according to Theorem 2.4, each Hankel matrix $H_{MN}(A, B, C)$, $M, N \geqslant n$, has the Bruhat decomposition

$$H_{MN}(A, B, C) = X_M P Y_N, \tag{3.2}$$

where P is an $n \times n$ permutation matrix, X_M is a $\mathscr{B}_P^-(n)$-row pattern matrix of size $Mp \times n$ with index list I, and Y_N is an $\mathscr{U}^+(n)$-column pattern matrix of size $n \times Nm$ with index list J.

Note that the Bruhat symbol (I, J), and hence the Bruhat permutation P, of $H_{MN}(A, B, C)$ is independent on the integers M and N, since I and J identify the indices of the first n linearly independent rows and columns of $H_{MN}(A, B, C)$, respectively (see Remark 2.6). This observation justifies the following definition.

Definition 3.1. The Bruhat symbol (respectively, Bruhat permutation) of the linear system $(A, B, C) \in \mathrm{L}_n^{p,m}$ is the Bruhat symbol (respectively, Bruhat permutation) of any Hankel matrix $H_{MN}(A, B, C)$, where $M, N \geqslant n$.

The Bruhat symbol has a straightforward interpretation in terms of controllability and observability indices, respectively. Let $(A, B, C) \in \mathrm{L}_n^{p,m}$ have Bruhat symbol (I, J). By parts (i) and (ii) of Lemma 2.8, both index lists I and J are nice. Hence, for $k = 1, \dots, p$ (respectively, $l = 1, \dots, m$), there exist smallest nonnegative integers α_k (respectively, β_l) such that $(k + p\alpha_k) \notin I$ (respectively, $(l + m\beta_l) \notin J$). Observe that $\sum_{k=1}^{p} \alpha_k = \sum_{l=1}^{m} \beta_l = n$ and that $\alpha = \alpha(C, A) := (\alpha_1, \dots, \alpha_p)$ and $\beta = \beta(A, B) := (\beta_1, \dots, \beta_m)$ coincide with the *output-* and *input-Kronecker indices* of (A, B, C), respectively. Note that in contrast to the familiar controllability (or observability) indices, the integers α_i are not monotonically ordered (the same for β_j). It is clear that $\alpha(C, A)$, $\beta(A, B)$ are uniquely determined by the Bruhat symbol of (A, B, C) and vice versa.

A systemtheoretic interpretation of the output- (or input-)Kronecker indices is that they are complete invariants for the group of restricted state feedback (or output injection) transformations; see [13]. The Bruhat permutation matrix has the interesting weaker feature of being invariant under restricted output feedback transformations. Let $\mathscr{G}$ be the group $\mathrm{GL}(n) \times \mathbb{R}^{m\times p} \times \mathscr{B}^-(p) \times \mathscr{B}^+(m)$, where the product is defined by

$$(T_1, K_1, U_1, V_1) \circ (T_2, K_2, U_2, V_2) := (T_1T_2, K_2 + V_2^{-1}K_1U_2, U_1U_2, V_1V_2).$$

Let us consider the following action of $\mathscr{G}$ on $\mathrm{L}_n^{p,m}$:

$$\begin{aligned}
&\mathscr{G} \times \mathrm{L}_n^{p,m} \to \mathrm{L}_n^{p,m},\\
&((T, K, U, V), (A, B, C)) \mapsto (T(A + BKC)T^{-1}, TBV^{-1}, UCT^{-1}).
\end{aligned}$$

The equivalence relation on $\mathrm{L}_n^{p,m}$, deduced from the above action, is referred to as the *restricted output feedback equivalence*. From the subsequent Lemma 3.2 it follows that the output- and input-Kronecker indices (or, equivalently, the Bruhat symbol) and the Bruhat permutation of $(A, B, C) \in \mathrm{L}_n^{p,m}$ are invariant under restricted output feedback equivalence. However, they are not all invariant under full output feedback equivalence.

Lemma 3.2. *Let $(A, B, C) \in \mathrm{L}_n^{p,m}$ have output- and input-Kronecker indices $\alpha(C, A)$ and $\beta(A, B)$, respectively, and Bruhat permutation $P = P(A, B, C)$. Then for all $(T, K, U, V) \in \mathrm{GL}(n) \times \mathbb{R}^{m\times p} \times \mathscr{B}^-(p) \times \mathscr{B}^+(m)$*

$$\begin{aligned}
\alpha(C, A) &= \alpha(UCT^{-1}, T(A + BKC)T^{-1}),\\
\beta(A, B) &= \beta(T(A + BKC)T^{-1}, TBV^{-1}),\\
P(A, B, C) &= P(T(A + BKC)T^{-1}, TBV^{-1}, UCT^{-1}).
\end{aligned}$$

Proof. One observes that the observability and controllability matrices have the following block structure:

$$O_M(UCT^{-1}, T(A+BKC)T^{-1}) = \underbrace{\begin{bmatrix} U & & 0 \\ & \ddots & \\ * & & U \end{bmatrix}}_{\text{lower triangular}} O_M(C, A)T^{-1},$$

$$R_N(T(A+BKC)T^{-1}, TBV^{-1}) = TR_N(A, B) \underbrace{\begin{bmatrix} V^{-1} & & * \\ & \ddots & \\ 0 & & V^{-1} \end{bmatrix}}_{\text{upper triangular}}.$$

Thus $H_{MN}(T(A+BKC)T^{-1}, TBV^{-1}, UCT^{-1}) = \widetilde{U} H_{MN}(A, B, C)\widetilde{V}$, where $\widetilde{U}$ and $\widetilde{V}$ are nonsingular lower and upper triangular matrices, respectively. This equation holds in particular for all $M, N \geqslant n$. Since the Bruhat symbol and the Bruhat permutation are invariant under multiplication on the left and on the right by nonsingular lower and upper triangular matrices, respectively, the result follows. □

Fixing the above three combinatorial data (output- and input-Kronecker indices, Bruhat permutation matrix) leads to a disjoint decomposition of the set of Hankel matrices into subsets. Assume that $M, N \geqslant n$ and $\max(M, N) > n$ and let $\mathrm{Hank}_{MN}(n, m, p)$ denote the set of all $M \times N$ block Hankel matrices $H = [H_{i+j-1}]_{i,j=1}^{M,N}$, $H_{i+j-1} \in \mathbb{R}^{p\times m}$, satisfying

$$\operatorname{rank} H = \operatorname{rank}[H_{i+j-1}]_{i,j=1}^{n} = n. \tag{3.3}$$

By Kalman's main realization theorem there exists for any block Hankel $H \in \mathrm{Hank}_{MN}(n, m, p)$ a controllable and observable system $(A, B, C) \in \mathrm{L}_n^{p,m}$, unique up to state space similarity, such that $H = H_{MN}(A, B, C)$.

Definition 3.3. Let M, N satisfy $M, N \geqslant n$ and $\max(M, N) > n$. Given any integer combinations $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_m)$ of n and any $n \times n$ permutation matrix P, let $\mathscr{B}(\alpha, \beta, P)$ denote the subset of all Hankel matrices $H = H_{MN}(A, B, C) \in \mathrm{Hank}_{MN}(n, m, p)$ such that (A, B, C) has output- and input-Kronecker indices $\alpha = \alpha(C, A)$, $\beta = \beta(A, B)$ and Bruhat permutation matrix $P = P(A, B, C)$. $\mathscr{B}(\alpha, \beta, P)$ is called a Bruhat stratum of $\mathrm{Hank}_{MN}(n, m, p)$.

The Bruhat strata thus define a decomposition of $\mathrm{Hank}_{MN}(n, m, p)$ into finitely many disjoint subsets. They do not need to be nonempty, though. In fact, from what is mentioned above we see that the union of strata

$$\mathscr{B}(\alpha, \beta) := \bigcup_{P \in \mathscr{W}(n)} \mathscr{B}(\alpha, \beta, P)$$

is nonempty for any pair α, β of combinations of n. However, this does not imply that $\mathscr{B}(\alpha, \beta, P)$ is nonempty for any permutation matrix P. The characterization of admissible permutation matrices of block Hankels $H = H_{MN}(A, B, C) \in \mathrm{Hank}_{MN}(n, m, p)$ is left as an open problem. The following remark clarifies the situation in the scalar case.

Remark 3.4. It follows from [18] (Theorem 2.3) that the Bruhat permutation of a scalar linear system $(A, b, c) \in \mathrm{L}_n^{1,1}$ is of the form

$$P = \mathrm{diag}[P_{\nu_1}, P_{\nu_2}, \dots, P_{\nu_r}], \tag{3.4}$$

where

$$P_{\nu_i} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & & 1 & 0 \\ \vdots & & & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}_{\nu_i \times \nu_i}, \quad i = 1, \dots, r,$$

and

$$\nu_1, \nu_1 + \nu_2, \dots, \nu_1 + \nu_2 + \cdots + \nu_r = n$$

are the orders of the nonzero leading principal minors of any associated Hankel matrix $H_{MN}(A, b, c)$, $M, N \geqslant n$. (The *leading principal minors* of a scalar $M \times N$ Hankel matrix $H = [h_{i+j-1}]$ are, by definition, the minors $\det[h_{i+j-1}]_{i,j=1}^{\mu}$, $\mu = 1, \dots, \min\{M, N\}$.) However, in the case when $\max\{p, m\} > 1$, a similar characterization of the Bruhat permutation is unknown.

4. Bruhat canonical form

The concept of Bruhat decomposition can be applied to obtain a new canonical form on $\mathrm{L}_n^{p,m}$ relative to similarity. The next theorem is instrumental in deriving the canonical form.

Theorem 4.1. *Let $(A, B, C) \in \mathrm{L}_n^{p,m}$ have Bruhat permutation P. With notation as in (3.2), there exists an unique nonsingular matrix T such that*

$$X_M P = O_M T^{-1} \quad \text{and} \quad Y_N = T R_N \tag{4.1}$$

whenever $M, N \geqslant n$. The matrix T does not depend on the integers M and N.

Proof. We first observe from (3.1) and (3.2) that

$$X_M P Y_N = O_M R_N, \quad M, N \geqslant n. \tag{4.2}$$

Since both X_M and Y_N have full rank n for all $M, N \geqslant n$, there are one-sided inverses $\widehat{X}_M$ and Y_N' such that $\widehat{X}_M X_M = Y_N Y_N' = I_n$ for all $M, N \geqslant n$, so that (4.2) becomes

$$I_n = (P^\top \widehat{X}_M O_M)(R_N Y_N'), \quad M, N \geqslant n. \tag{4.3}$$

Applying (4.3) with $N = n$, we also obtain that $I_n = (P^\top \widehat{X}_M O_M)(R_n Y_n')$ for all $M \geqslant n$, and hence $R_N Y_N' = R_n Y_n'$ for all $N \geqslant n$. Then with

$$T := (R_n Y_n')^{-1} = P^\top \widehat{X}_n O_n = P^\top \widehat{X}_M O_M = (R_N Y_N')^{-1}, \quad M, N \geqslant n$$

(4.1) follows. To prove uniqueness, suppose that $Y_k = S R_k = T R_k$ for some nonsingular matrices S, T and some integer $k \geqslant n$. Then clearly $(S - T) R_k = 0$ and, since R_k has a right inverse, it turns out that $S = T$. $\square$

Corollary 4.2. *The mapping $(A, B, C) \in \mathrm{L}_n^{p,m} \mapsto (TAT^{-1}, TB, CT^{-1})$ is a canonical form for the similarity relation, where T is defined by (4.1).*

Proof. Suppose that $(\widetilde{A}, \widetilde{B}, \widetilde{C}) = (SAS^{-1}, SB, CS^{-1})$ for some nonsingular matrix S. By Theorem 4.1, there are unique nonsingular matrices T and $\widetilde{T}$ such that $Y_n = T R_n(A, B)$ and $Y_n =$

$\widetilde{T}R_n(\widetilde{A},\widetilde{B})$, respectively. It follows that $TR_n(A,B)=\widetilde{T}R_n(\widetilde{A},\widetilde{B})=\widetilde{T}SR_n(A,B)$ and, since $R_n(A,B)$ is right invertible, this implies that $T=\widetilde{T}S$. Thus, $(\widetilde{T}\widetilde{A}\widetilde{T}^{-1},\widetilde{T}\widetilde{B},\widetilde{C}\widetilde{T}^{-1})=(TAT^{-1},TB,CT^{-1})$. □

Definition 4.3. $(\widehat{A},\widehat{B},\widehat{C}):=(TAT^{-1},TB,CT^{-1})$, where T is defined by (4.1), is referred to as the Bruhat canonical form of $(A,B,C)\in \mathrm{L}_n^{p,m}$.

The following characterization of the Bruhat canonical form in terms of the $(n+1)$th order observability and controllability matrices is an immediate consequence of Theorems 2.4 and 4.1.

Theorem 4.4. *Let $(A,B,C)\in \mathrm{L}_n^{p,m}$ have Bruhat symbol (I,J) and Bruhat permutation P. The Bruhat canonical form $(\widehat{A},\widehat{B},\widehat{C})$ of (A,B,C) is uniquely determined by the following conditions*:

(i) *$(\widehat{A},\widehat{B},\widehat{C})$ is similar to (A,B,C);*
(ii) *$R_{n+1}(\widehat{A},\widehat{B})$ is an $\mathcal{U}^+(n)$-column pattern matrix with index list J;*
(iii) *$O_{n+1}(\widehat{C},\widehat{A})$ is a $\mathcal{B}^-(n)P$-row pattern matrix with index list I;*
(iv) *$O_{n+1}(\widehat{C},\widehat{A})$ is a $P\mathcal{B}^-(n)$-row pattern matrix with index list I.*

Proof. Let $H=XPY$ be the Bruhat decomposition of the Hankel matrix $H:=H_{(n+1)(n+1)}\times(A,B,C)$ (see Theorem 2.4). By Theorem 4.1 and Definition 4.3, $XP=O_{n+1}(\widehat{C},\widehat{A})$ and $Y=R_{n+1}(\widehat{A},\widehat{B})$. Thus, the result follows. □

We now ask whether conditions (ii)–(iv) of Theorem 4.4 can be translated into equivalent conditions in terms of $(\widehat{A},\widehat{B},\widehat{C})$. To answer the question we need a lemma, which says that left (respectively, right) factors in full rank factorizations of Hankel matrices, whose Bruhat symbols satisfy condition (4.4), are observability (respectively, controllability) matrices.

Lemma 4.5. *Let $H\in \mathrm{Hank}(n,Mp\times Nm)$ have the Bruhat symbol $(I,J)=(\{i_1,\dots,i_n\},\{j_1,\dots,j_n\})$, and assume that*

$$i_n \leqslant (M-1)p,\qquad j_n\leqslant (N-1)m. \tag{4.4}$$

Let H have full rank factorization $H=UV$. Then

$$U=O_M(C,A),\qquad V=R_N(A,B), \tag{4.5}$$

where

$$A=U_I^{-1}U_{I+p}=V^{J+m}(V^J)^{-1},\qquad B=V^{\underline{m}},\qquad C=U_{\underline{p}}, \tag{4.6}$$

and $\underline{p}:=\{1,2,\dots,p\}$, $\underline{m}:=\{1,2,\dots,m\}$.

Proof. Recall the notation introduced after Theorem 2.1. By the block Hankel form of $H=UV$ and (4.4) we have

$$U_{I+p}V^J=U_IV^{J+m} \tag{4.7}$$

and

$$\begin{aligned}U_{\underline{p}+kp}V^J&=U_{\underline{p}+(k-1)p}V^{J+m},\quad k=1,\dots,M-1,\\ U_IV^{\underline{m}+lm}&=U_{I+p}V^{\underline{m}+(l-1)m},\quad l=1,\dots,N-1,\end{aligned} \tag{4.8}$$

where $\underline{p}+kp$ and $\underline{m}+lm$ are short for $\{1+kp,2+kp,\dots,p+kp\}$ and $\{1+lm,2+lm,\dots,m+lm\}$, respectively. Eq. (4.7) implies $U_I^{-1}U_{I+p}=V^{J+m}(V^J)^{-1}$. It follows from (4.8) that

$$U_{\underline{p}+kp} = U_{\underline{p}}[V^{J+m}(V^J)^{-1}]^k, \quad k = 0, 1, \ldots, M-1,$$
$$V^{\underline{m}+lm} = [U_I^{-1}U_{I+p}]^l V^{\underline{m}}, \quad l = 0, 1, \ldots, N-1$$

by induction for k and l. Then $U_I^{-1}U_{I+p} = V^{J+m}(V^J)^{-1}$ readily implies (4.5) and (4.6). $\square$

If condition (4.4) is not satisfied, then Lemma 4.5 does not hold.

Example 4.6 ($p = m = n = M = 2,\ N = 3$). Consider the full rank factorization

$$H = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Condition (4.4) is not satisfied, since H has Bruhat symbol $(\{1, 4\}, \{1, 6\})$. However, there is no pair (C, A) of 2×2 matrices satisfying

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = O_2(C, A).$$

To simplify the notation, let us in the sequel use the notations

$$\begin{aligned} &H(A, B, C) := H_{(n+1)(n+1)}(A, B, C), \\ &O := O(C, A) := O_{n+1}(C, A) \quad \text{and} \quad R := R(A, B) := R_{n+1}(A, B). \end{aligned} \tag{4.9}$$

Let $(A, B, C) \in \mathrm{L}_n^{p,m}$ have the Bruhat symbol (I, J). Since I and J are nice, it follows that the index lists $\underline{p} \cup (I + p)$ and $\underline{m} \cup (J + m)$ also are nice, where $\underline{p} := \{1, 2, \ldots, p\}$ and $\underline{m} := \{1, 2, \ldots, m\}$. Therefore

$$I \subset \underline{p} \cup (I + p), \qquad J \subset \underline{m} \cup (J + m). \tag{4.10}$$

With notation as in (4.9), we observe from (4.10) that $\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}$ contains O_I (respectively, $[R^{\underline{m}},\quad R^{J+m}]$ contains R^J) as a submatrix. Consequently, there are unique index lists $I' = \{i'_1, \ldots, i'_n\}$ and $J' = \{j'_1, \ldots, j'_n\}$ with $i'_n \leqslant n + p - 1$ and $j'_n \leqslant n + m - 1$ satisfying

$$\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}_{I'} = O_I, \qquad [R^{\underline{m}} \quad R^{J+m}]^{J'} = R^J. \tag{4.11}$$

Definition 4.7. I' and J', respectively, are referred to as the row and column successor index lists of $(A, B, C) \in \mathrm{L}_n^{p,m}$.

Remark 4.8. (i) (I, J) as well as I' and J' are invariant under similarity.

(ii) I' indicates within the set of $p + n$ rows of $O(C, A)$ indexed by $1, \ldots, p, i_1 + p, \ldots, i_n + p$ the first n linearly independent rows. An analogous statement holds for J'.

(iii) I' can be constructed from I as follows. Identify I with the output-Kronecker indices $(\alpha_1, \ldots, \alpha_p)$. Consider an array of p columns and put an $\times$ in the first α_k positions and an $\bullet$ in the position $(\alpha_k + 1)$ of column k, $k = 1, \ldots, p$. Form a single row of length $p + n$ by concatenating the successive rows and omitting the empty positions. Then the lth $\times$ in this row has position i'_l, $l = 1, \dots, n$. By inverting this procedure, I can be constructed from I'. Analogous statements holds for J' and J. Thus, the map $(I, J) \mapsto (I', J')$ is bijective.

Example 4.9. Let $p = 4$, $n = 7$ and $I = \{1, 3, 4, 5, 7, 11, 15\}$, that is, $\alpha = (2, 0, 4, 1)$. Form an array of $p = 4$ columns as

$$\begin{array}{cccc} \times & \bullet & \times & \times \\ \times & & \times & \bullet \\ \bullet & & \times & \\ & & \times & \\ & & \bullet & \end{array}$$

and concatenation of its rows gives $\times \bullet \times \times \times \times \bullet \bullet \times \times \bullet$. The positions of the $\times$ in this sequence constitute $I' = \{1, 3, 4, 5, 6, 9, 10\}$.

We are now ready to express conditions (ii) and (iii) of Theorem 4.4 by equivalent conditions in terms of $(\widehat{A}, \widehat{B}, \widehat{C})$.

Proposition 4.10. *Let $(A, B, C) \in \mathrm{L}_n^{p,m}$ have Bruhat symbol (I, J), row and column successor index lists I' and J', respectively, and Bruhat permutation P. Then*

(i) *$R(A, B)$ is an $\mathscr{U}^+(n)$-column pattern matrix with index list J if and only if $[B,\ A]$ is an $\mathscr{U}^+(n)$-column pattern matrix with index list J'.*
(ii) *$O(C, A)$ is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I if and only if $\begin{bmatrix} C \\ PA \end{bmatrix}$ is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I'.*

Proof. With the notation as in (4.9), it follows from Lemma 4.5 that $A = O_I^{-1} O_{I+p} = R^{J+m}(R^J)^{-1}$, $B = R^{\underline{m}}$ and $C = O_{\underline{p}}$. Hence

$$\begin{bmatrix} C \\ PA \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & PO_I^{-1} \end{bmatrix} \begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}, \quad \begin{bmatrix} B & A \end{bmatrix} = \begin{bmatrix} R^{\underline{m}} & R^{J+m} \end{bmatrix} \begin{bmatrix} I_m & 0 \\ 0 & (R^J)^{-1} \end{bmatrix}, \tag{4.12}$$

where I_p and I_m denote the identity matrices of sizes $p \times p$ and $m \times m$, respectively.

(ii) Suppose that O is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I. Since each row $i \notin I$ of O is linearly dependent on rows $1, \dots, i-1$, we deduce by use of (4.11) that each row $i \notin I'$ of $\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}$ is linearly dependent on rows $1, \dots, i-1$ of $\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}$. Further, $O_I \in \mathscr{B}^-(n)P$ implies that

$$\begin{bmatrix} I_p & 0 \\ 0 & PO_I^{-1} \end{bmatrix} \in \mathscr{B}^-(p+n). \tag{4.13}$$

Multiplying $\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}$ on the left by a nonsingular lower triangular matrix leaves rank and linear (in)dependencies amongst the rows of $\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}$ invariant. Hence, it follows from (4.12) and (4.13) that each row $i \notin I'$ of $\begin{bmatrix} C \\ PA \end{bmatrix}$ is linearly dependent on rows $1, \dots, i-1$. Again using (4.12), (4.13) and the fact that each row $i \notin I'$ of $\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}$ is linearly dependent on rows $1, \ldots, i-1$, we see that, for all $k = 1, \ldots, n$, the kth row of $\begin{bmatrix} C \\ PA \end{bmatrix}_{I'}$ is the sum of a nonzero multiple of row k of $\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}_{I'}$ and a linear combination of the rows $1, \ldots, k-1$ of $\begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}_{I'}$. Consequently, there is a matrix $S \in \mathscr{B}^-(n)$ such that

$$\begin{bmatrix} C \\ PA \end{bmatrix}_{I'} = S \begin{bmatrix} O_{\underline{p}} \\ O_{I+p} \end{bmatrix}_{I'},$$

so that (4.11) gives $\begin{bmatrix} C \\ PA \end{bmatrix}_{I'} = SO_I \in \mathscr{B}^-(n)P$.

Conversely, suppose that $\begin{bmatrix} C \\ PA \end{bmatrix}$ is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I'. Since $\begin{bmatrix} CP^\top \\ PAP^\top \end{bmatrix}_{I'} \in \mathscr{B}^-(n)$ and each row $i \notin I'$ of $\begin{bmatrix} CP^\top \\ PAP^\top \end{bmatrix}$ is linearly dependent on rows $1, \ldots, i-1$, it follows that the first n linearly independent rows of $O(C, A)P^\top$ constitute a lower triangular matrix with nonzero diagonal entries. Thus, $O(C, A)$ is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I.

(i) is shown similarly as (ii); details are left to the reader. □

Proposition 4.10 leaves open the question whether it is possible to translate condition (iv) of Theorem 4.4 into an equivalent condition in terms of $(\widehat{C}, \widehat{A})$. Despite intensive efforts, we have not been able to answer this question. Applying Proposition 4.10 we obtain the following reformulation of Theorem 4.4.

Theorem 4.11 (Bruhat canonical form). *Let $(A, B, C) \in \mathrm{L}_n^{p,m}$ be given with Bruhat symbol (I, J), row and column successor index lists I' and J', respectively, and Bruhat permutation P. There exists an unique $(\widehat{A}, \widehat{B}, \widehat{C})$ satisfying*

(i) *$(\widehat{A}, \widehat{B}, \widehat{C})$ is similar to (A, B, C);*
(ii) *$[\widehat{B}\ \widehat{A}]$ is an $\mathscr{U}^+(n)$-column pattern matrix with index list J';*
(iii) *$\begin{bmatrix} \widehat{C} \\ P\widehat{A} \end{bmatrix}$ is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I';*
(iv) *$O(\widehat{C}, \widehat{A})$ is a $P\mathscr{B}^-(n)$-row pattern matrix with index list I.*

Moreover, the map

$$(A, B, C) \mapsto (\widehat{A}, \widehat{B}, \widehat{C})$$

is a complete invariant for state space similarity on $\mathrm{L}_n^{p,m}$.

The next lemma provides a usable criterion for a matrix to be contained in $\mathscr{B}^-(n)P$ (respectively, $P\mathscr{B}^-(n)$).

Lemma 4.12. *Let $S = [s_{ij}]$ be a nonsingular $n \times n$ matrix, and let P be an $n \times n$ permutation matrix with the 1's at the positions $(1, p_1), \ldots, (n, p_n)$ and zeros elsewhere. Then $S \in \mathscr{B}^-(n)P$ (respectively, $S \in P\mathscr{B}^-(n)$) if and only if, for all $i = 1, \ldots, n$, the p_ith column (respectively, ith row) of S is of the form*

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ s_{ip_i} \\ \vdots \\ s_{np_i} \end{bmatrix} \quad (\textit{respectively}, \quad [s_{i1} \quad \cdots \quad s_{ip_i} \quad 0 \quad \cdots \quad 0]), \quad s_{ip_i} \neq 0.$$

Proof. The proof relies on the fact that, for all $i = 1, \dots, n$, column i of $SP^\top$ (respectively, row p_i of $P^\top S$) and column p_i (respectively, row i) of S coincide. □

In the case $p = m = 1$ the structure of the Bruhat canonical form is as follows.

Corollary 4.13. *Given $(A, b, c) \in \mathrm{L}_n^{1,1}$, let $\nu_1, \nu_1 + \nu_2, \dots, \nu_1 + \nu_2 + \cdots + \nu_r = n$ be the orders of the nonzero leading principal minors of $H(A, b, c)$ (see Remark 3.4). The Bruhat canonical form $(\widehat{A}, \hat{b}, \hat{c})$ of (A, b, c) has the shape*

$$(\widehat{A}, \hat{b}, \hat{c}) = \left(\begin{bmatrix} \widehat{A}_1 & N_1' & & & \\ N_1 & \widehat{A}_2 & N_2' & & \\ & N_2 & \widehat{A}_3 & & \\ & & \ddots & \ddots & N_{r-1}' \\ & & & N_{r-1} & \widehat{A}_r \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, [0, \dots, 0, \gamma, 0, \dots, 0] \right). \tag{4.14}$$

Here γ is a nonzero scalar at position ν_1, $\widehat{A}_i$ are the $\nu_i \times \nu_i$ matrices

$$\widehat{A}_1 = \begin{bmatrix} a_1^{(1)} & a_2^{(1)} & \cdots & & a_{\nu_1}^{(1)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix},$$

$$\widehat{A}_i = \begin{bmatrix} a_1^{(i)} & a_2^{(i)} & \cdots & & a_{\nu_i}^{(i)} \\ 1 & 0 & \cdots & 0 & p_{\nu_i-1}^{(i)} \\ 0 & 1 & & 0 & p_{\nu_i-2}^{(i)} \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & p_1^{(i)} \end{bmatrix}, \quad 2 \leqslant i \leqslant r, \tag{4.15}$$

$p_k^{(i)} := p_k^{(i)}(a_1^{(1)}, \dots, a_{\nu_1}^{(1)}, \dots, a_1^{(i-1)}, \dots, a_{\nu_{i-1}}^{(i-1)})$, $k = 1, \dots, \nu_i - 1$, are polynomials in the variables $a_1^{(1)}, \dots, a_{\nu_1}^{(1)}, \dots, a_1^{(i-1)}, \dots, a_{\nu_{i-1}}^{(i-1)}$, and N_i' and N_i are matrices (of appropriate dimensions), each of which has zeros in all positions except the northeast corner, where for N_i' there is a nonzero scalar and for N_i a one.

Proof. First, we observe that $I = J = \{1, 2, \dots, n\}$. Thus, $I' = J' = \{1, 2, \dots, n\}$ and condition (ii) of Theorem 4.11 provides $\hat{b} = [1, 0, \dots, 0]^\top$. By Remark 3.4, the Bruhat permutation of

(A, b, c) is of the form (3.4). Applying Lemma 4.12 and condition (iii) of Theorem 4.11, it follows that $\hat{c} = [0, \dots, 0, \gamma, 0, \dots, 0]$, where $\gamma \neq 0$ is at position ν_1. We deduce from Lemma 4.12 and conditions (ii) and (iii) of Theorem 4.11 that

$$\widehat{A} = \begin{bmatrix} \widetilde{A}_1 & N_1' & & & \\ N_1 & \widetilde{A}_2 & N_2' & & \\ & N_2 & \widetilde{A}_3 & & \\ & & \ddots & \ddots & N_{r-1}' \\ & & & N_{r-1} & \widetilde{A}_r \end{bmatrix},$$

$$\text{where } \widetilde{A}_i = \begin{bmatrix} \times & \times & \times & \cdots & \times & \times \\ 1 & \times & \times & & \times & \times \\ 0 & 1 & \times & & \times & \times \\ 0 & 0 & 1 & & \times & \times \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \times \end{bmatrix}, \quad i = 1, \dots, r, \tag{4.16}$$

and $\times$ represents an arbitrary entry of $\widetilde{A}_i$. Finally, using (4.16), Lemma 4.12 and part (iv) of Theorem 4.11, it is found that $\widetilde{A}_i = \widehat{A}_i$, $i = 1, \dots, r$. $\square$

The above Bruhat canonical form for scalar systems coincides, up to minor modifications, with the canonical form, proposed by Kalman [23] via continued fraction expansions. In fact, Kalman's canonical form differs from the above form in three details: (a) all coefficients $p_j^{(i)}$ are zero, (b) the parameters in $\widehat{A}_i$ appear in the last column instead of the first row, and (c) the parameters in the upper and lower blocks are rescaled. All this can be achieved by simple linear state space coordinate transformations.

We close this section by two toy examples that illustrate the computation of the Bruhat canonical form for a given matrix triple $(A, B, C) \in \mathrm{L}_n^{p,m}$. Of course, no attempt for efficient numerical computations is made.

Examples 4.14. (a) ($n = 5$, $p = m = 1$). Let

$$(A, b, c) = \left(\begin{bmatrix} -15 & 13 & -2 & 1 & -3 \\ -45 & 30 & -2 & 10 & -12 \\ -22 & 23 & -2 & -2 & -4 \\ -34 & 26 & -1 & 4 & -8 \\ -48 & 29 & 2 & 12 & -14 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix}^{\top} \right) \in \mathrm{L}_5^{1,1}.$$

Then (see (4.9))

$$H := H_{66}(A, b, c) = \begin{bmatrix} \frac{1}{2} & -1 & 2 & -4 & 8 & -13 \\ -1 & 2 & -4 & 8 & -13 & 35 \\ 2 & -4 & 8 & -13 & 35 & -46 \\ -4 & 8 & -13 & 35 & -46 & 128 \\ 8 & -13 & 35 & -46 & 128 & -256 \\ -13 & 35 & -46 & 128 & -256 & 107 \end{bmatrix}$$

has the Bruhat decomposition

$$H = \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 \\ -4 & 0 & 0 & 3 & 0 \\ 8 & 0 & 0 & 0 & 3 \\ -13 & -3 & -27 & -3 & 9 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & -2 & 4 & -8 & 16 & -26 \\ 0 & 1 & 1 & 6 & 0 & -16 \\ 0 & 0 & 1 & 1 & 6 & 8 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}}_{Y}$$

and Bruhat symbol $(I, J) = (\{1, 2, 3, 4, 5\}, \{1, 2, 3, 4, 5\})$. It follows from Lemma 4.5 (see (4.6)) that (A, b, c) has the Bruhat canonical form

$$\begin{aligned}(\widehat{A}, \hat{b}, \hat{c}) &\overset{(4.6)}{=} (Y^{\{2,3,4,5,6\}}(Y^J)^{-1}, Y^{\{1\}}, X_{\{1\}}P) \\ &= \left(\begin{bmatrix} -2 & 4 & -8 & 16 & -26 \\ 1 & 1 & 6 & 0 & -16 \\ 0 & 1 & 1 & 6 & 8 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -6 & 2 & 18 \\ 0 & 1 & -1 & -5 & 11 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top P \right) \\ &= \left(\begin{bmatrix} -2 & 0 & 0 & 0 & 6 \\ 1 & 3 & -1 & -9 & -17 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2}, & 0, & 0, & 0, & 0 \end{bmatrix} \right).\end{aligned}$$

(b) $(n = 3, p = m = 2)$. Let

$$(A, B, C) = \left(\begin{bmatrix} -173 & 217 & -249 \\ -48 & 60 & -68 \\ 82 & -103 & 119 \end{bmatrix}, \begin{bmatrix} -12 & 11 \\ 2 & 3 \\ 10 & -5 \end{bmatrix}, \begin{bmatrix} 5 & -6 & 7 \\ 15 & -18 & 21 \end{bmatrix} \right) \in \mathrm{L}_3^{2,2}.$$

Then (see (4.9))

$$H := H_{44}(A, B, C) = \begin{bmatrix} \begin{bmatrix} -2 & 2 \\ -6 & 6 \end{bmatrix} & \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} & \begin{bmatrix} 4 & -3 \\ 12 & -9 \end{bmatrix} & \begin{bmatrix} -4 & 1 \\ -12 & 3 \end{bmatrix} \\ \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} & \begin{bmatrix} 4 & -3 \\ 12 & -9 \end{bmatrix} & \begin{bmatrix} -4 & 1 \\ -12 & 3 \end{bmatrix} & \begin{bmatrix} -4 & 7 \\ -12 & 21 \end{bmatrix} \\ \begin{bmatrix} 4 & -3 \\ 12 & -9 \end{bmatrix} & \begin{bmatrix} -4 & 1 \\ -12 & 3 \end{bmatrix} & \begin{bmatrix} -4 & 7 \\ -12 & 21 \end{bmatrix} & \begin{bmatrix} 20 & 15 \\ 60 & 45 \end{bmatrix} \\ \begin{bmatrix} -4 & 1 \\ -12 & 3 \end{bmatrix} & \begin{bmatrix} -4 & 7 \\ -12 & 21 \end{bmatrix} & \begin{bmatrix} 20 & 15 \\ 60 & 45 \end{bmatrix} & \begin{bmatrix} 100 & 77 \\ 300 & 231 \end{bmatrix} \end{bmatrix}$$

has the Bruhat decomposition

$$H = \underbrace{\begin{bmatrix} -2 & 0 & 0 \\ -6 & 0 & 0 \\ 4 & 3 & 0 \\ 12 & 9 & 0 \\ 4 & 1 & \frac{2}{3} \\ 12 & 3 & 2 \\ -4 & -3 & 4 \\ -12 & -9 & 12 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & -1 & -2 & \frac{1}{2} & -2 & \frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & 1 & 4 & -\frac{5}{3} & \frac{4}{3} & -\frac{5}{3} & -4 & 3 \\ 0 & 0 & 0 & 1 & 4 & 4 & 24 & 21 \end{bmatrix}}_{Y}$$

and Bruhat symbol $(I, J) = (\{1, 3, 5\}, \{1, 2, 4\})$. By Lemma 4.5, we conclude that the Bruhat canonical form of (A, B, C) is

$$\begin{aligned}(\widehat{A}, \widehat{B}, \widehat{C}) &\overset{(4.6)}{=} (Y^{\{3,4,6\}}(Y^J)^{-1}, Y^{\{1,2\}}, X_{\{1,2\}}P) \\ &= \left(\begin{bmatrix} -2 & \frac{1}{2} & \frac{3}{2} \\ 4 & -\frac{5}{3} & -\frac{5}{3} \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & \frac{7}{6} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 \\ -6 & 0 & 0 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} -2 & -\frac{3}{2} & 0 \\ 4 & \frac{7}{3} & \frac{2}{9} \\ 0 & 1 & \frac{17}{3} \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 \\ -6 & 0 & 0 \end{bmatrix} \right). \end{aligned} \tag{4.17}$$

5. Bruhat versus Bosgra–van der Weiden canonical form

Using the concept of a Γ-row (respectively, Γ-column) pattern matrix, developed in Section 2, the Bosgra–van der Weiden canonical form can be described as follows.

Theorem 5.1 (Bosgra and van der Weiden [3]). *Let $(A, B, C) \in \mathrm{L}_n^{p,m}$ have row and column successor index lists I' and J', respectively. There exist an unique $n \times n$ permutation matrix P and an unique $(\widetilde{A}, \widetilde{B}, \widetilde{C}) \in \mathrm{L}_n^{p,m}$ satisfying*

(a) *conditions* (i)–(iii) *of Theorem* 4.11;
(b) $\begin{bmatrix} \widetilde{C} \\ P\widetilde{A} \end{bmatrix}$ *is a $P\mathscr{B}^-(n)$-row pattern matrix with index list I'.*

The matrix triplet $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is known as the *Bosgra–van der Weiden canonical form* of $(A, B, C) \in \mathrm{L}_n^{p,m}$. There is a fundamental constraint on the set of state space coordinate transformations that transform a realization in Bruhat form into one in Bosgra–van der Weiden form. To clarify this, we prove that the unique matrix that reduces a Bruhat canonical matrix triplet to its Bosgra–van der Weiden canonical matrix triplet necessarily belongs to the group $\mathscr{U}^+(n) \cap P^\top \mathscr{U}^-(n) P$.

Lemma 5.2. *Let $(\widehat{A}, \widehat{B}, \widehat{C}) \in \mathrm{L}_n^{p,m}$ be in Bruhat canonical form with row and column successor index lists I' and J', respectively. For any nonsingular $n \times n$ matrix T let*

$$(\widehat{A}_T, \widehat{B}_T, \widehat{C}_T) := (T\widehat{A}T^{-1}, T\widehat{B}, \widehat{C}T^{-1}).$$

Then

(i) $[\widehat{B}_T \quad \widehat{A}_T]$ *is an* $\mathscr{U}^+(n)$*-column pattern matrix with index list* J' *if and only if* $T \in \mathscr{U}^+(n)$,

(ii) $\begin{bmatrix} \widehat{C}_T \\ P\widehat{A}_T \end{bmatrix}$ *is a* $\mathscr{B}^-(n)P$*-row pattern matrix with index list* I' *if and only if* $T \in P^\top\mathscr{B}^-(n)P$.

Proof. Let (I, J) be the Bruhat symbol of $(\widehat{A}, \widehat{B}, \widehat{C})$.

(ii) First, if $\begin{bmatrix} \widehat{C}_T \\ P\widehat{A}_T \end{bmatrix}$ is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I', by part (ii) of Proposition 4.10, $O(\widehat{C}_T, \widehat{A}_T)$ is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I (see also part (i) of Remark 4.8). Thus

$$O(\widehat{C}_T, \widehat{A}_T)_I = O(\widehat{C}, \widehat{A})_I T^{-1} \in \mathscr{B}^-(n)P. \tag{5.1}$$

Observe that, by part (iii) of Theorem 4.4,

$$O(\widehat{C}, \widehat{A})_I \in \mathscr{B}^-(n)P. \tag{5.2}$$

Using (5.1) and (5.2), we find that $T^{-1} \in P^\top\mathscr{B}^-(n)P$.

Conversely, if $T \in P^\top\mathscr{B}^-(n)P$, then $PT^{-1}P^\top \in \mathscr{B}^-(n)$ and, for all $k = 1, \dots, n$, the kth column of $O(\widehat{C}, \widehat{A})P^\top(PT^{-1}P^\top)$ is the sum of a nonzero multiple of column k of $O(\widehat{C}, \widehat{A})P^\top$ and a linear combination of the columns $k+1, \dots, n$ of $O(\widehat{C}, \widehat{A})P^\top$. Thus multiplication of $O(\widehat{C}, \widehat{A})P^\top$ by $PT^{-1}P^\top$ on the right leaves the $\mathscr{B}^-(n)$-row pattern structure with index list I of $O(\widehat{C}, \widehat{A})P^\top$ (see part (iii) of Theorem 4.4) invariant. Consequently, $O(\widehat{C}_T, \widehat{A}_T)$ is a $\mathscr{B}^-(n)P$-row pattern matrix with index list I. The result now follows from part (ii) of Proposition 4.10.

A similar proof also yields (i). □

We note that

$$\mathscr{U}^+(n) \cap P^\top\mathscr{B}^-(n)P = \mathscr{U}^+(n) \cap P^\top\mathscr{U}^-(n)P.$$

This implies the following corollary, which shows that the structure indices appearing in the Bosgra–van der Weiden form coincide with the Bruhat symbol I, J and Bruhat permutation matrix P. In particular, fixing the structural invariants in the Bruhat canonical form and the Bosgra–van der Weiden form yields identical subsets of $\mathrm{Hank}_{MN}(n, m, p)$, i.e. the Bruhat strata.

Corollary 5.3. *Given* $(\widehat{A}, \widehat{B}, \widehat{C}) \in \mathrm{L}_n^{p,m}$ *in Bruhat canonical form. With notation as in Lemma 5.2, let* T *be the unique nonsingular matrix such that* $(\widehat{A}_T, \widehat{B}_T, \widehat{C}_T)$ *is in Bosgra–van der Weiden canonical form. Then* $T \in \mathscr{U}^+(n) \cap P^\top\mathscr{U}^-(n)P$. *In particular, if* $P = I_n$*, then the Bosgra–van der Weiden form coincides with the Bruhat canonical form.*

We see that there is a remarkable relationship between the Bosgra–van der Weiden and the Bruhat canonical form. Except for condition (b) of Theorem 5.1, which is not equivalent to condition (iv) of Theorem 4.11, the structure of the Bosgra–van der Weiden canonical form is determined by the same conditions as the structure of the Bruhat canonical form. The following example explains the computation of the Bosgra–van der Weiden canonical form for the system matrices considered in Examples 4.14.

Example 5.4. (a) Let $(A, b, c) \in \mathrm{L}_5^{1,1}$ be as in Example 4.14(a). Then $I' = J' = \{1, 2, 3, 4, 5\}$ are the row and column successor index lists of (A, b, c). Using the conditions of Theorem 5.1, a straightforward computation shows that

$$(\widetilde{A}, \tilde{b}, \tilde{c}) = \left(\begin{bmatrix} -2 & 0 & 0 & 0 & 6 \\ 1 & 5 & -11 & 13 & -27 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

is the Bosgra–van der Weiden canonical form of (A, b, c). Note, that the second row and last column of $\widetilde{A}$ differs from those in $\widehat{A}$, appearing in Example 4.14(a). This also shows the general property, that the Bosgra–van der Weiden canonical form differs at least from the Bruhat canonical form (4.14) and (4.15), in so far that all coefficients $p_j^{(i)}$ (cf. (4.15)) are set to zero.

(b) Let $(A, B, C) \in \mathrm{L}_3^{2,2}$ be as in Example 4.14(b). In this case the Bruhat permutation of (A, B, C) is the identity matrix. Hence, according to Corollary 5.3, the Bosgra–van der Weiden canonical form of $(A, B, C) \in \mathrm{L}_3^{2,2}$ coincides with the Bruhat canonical form (4.17).

6. Bruhat cell decomposition

In this section we show that the above results can be reinterpreted as defining a cell decomposition of the space of linear systems. We use the following notation. Let $\mathrm{Rat}(n, m, p)$ denote the set of real rational, strictly proper $p \times m$ transfer functions $G(s) \in \mathbb{R}(s)^{p\times m}$ of McMillan degree n. For any integers M, N with $M, N \geqslant n$, $\max(M, N) > n$ let $\mathrm{Hank}_{MN}(n, m, p)$ denote the set of $M \times N$ block Hankel matrices $H = [H_{i+j-1}]_{i,j=1}^{M,N}$, $H_{i+j-1} \in \mathbb{R}^{p\times m}$, satisfying the rank condition (3.3). It is well known that $\mathrm{Rat}(n, m, p)$ and $\mathrm{Hank}_{MN}(n, m, p)$ are smooth manifolds of dimension $n(m + p)$ and that the map

$$\rho : \mathrm{Rat}(n, m, p) \to \mathrm{Hank}_{MN}(n, m, p)$$

that assigns to every transfer function G with Laurent expansion

$$G(s) = \sum_{r=1}^{\infty} H_r s^{-r}$$

the block Hankel matrix $\rho(G) = [H_{i+j-1}]_{i,j=1}^{M,N}$ is a diffeomorphism. For $m = p = 1$ we use the shortened notation $\mathrm{Rat}(n)$ and $\mathrm{Hank}_{M,N}(n)$ for $\mathrm{Rat}(n, 1, 1)$ and $\mathrm{Hank}_{MN}(n, 1, 1)$, respectively. In [11] a decomposition of $\mathrm{Rat}(n)$ into disjoint cells was introduced using the continued fraction expansion of transfer functions. The image via the diffeomorphism ρ of such cells then induces an equivalent cell decomposition of the space of Hankels $\mathrm{Hank}_{M,N}(n)$; we refer to this cell decomposition as the *Fuhrmann–Krishnaprasad cell decomposition* of $\mathrm{Hank}_{M,N}(n)$. For multivariable systems, Antoulas and Bishop [1] have proposed a generalization of the Kalman canonical form via matrix continued fraction expansions. However, they did not inspect the possibility whether or not this canonical form actually induces a cell decomposition of $\mathrm{Rat}(n, m, p)$. Thus, a generalization of the Fuhrmann–Krishnaprasad cell decomposition to matrix-valued transfer functions is unknown.

In order to construct such a decomposition, we proceed as follows. Let $\mathscr{B}(\alpha, \beta, P)$ denote the Bruhat stratum, defined by fixing the Bruhat symbol (α, β) (or, equivalently, the output- and input-Kronecker indices) and Bruhat permutation P, respectively. Since $\mathscr{B}(\alpha, \beta, P)$ is characterized by the conditions (a) and (b) in the Bosgra–van der Weiden form (Theorem 5.1) we see, that $\mathscr{B}(\alpha, \beta, P)$ decomposes into a finite union of relatively open, disjoint cells

$$\mathscr{B}(\alpha, \beta, P) = \bigcup_i C_i(\alpha, \beta, P).$$

We refer to this as the *Bruhat cell decomposition* of $\mathrm{Hank}_{MN}(n, m, p)$. There is a partial order on the Bruhat strata that induces a partial order on the Bruhat cells that is defined as follows. Let $\alpha \prec_K \alpha'$ and $\beta \prec_K \beta'$ denote the Kronecker orders on combinations, as introduced in [13]. Moreover, on the set of permutation matrices $\mathscr{W}(n)$ we consider the partial ordering

$$P \prec_w P' \iff \text{rank } P_{rs} \leqslant \text{rank } P'_{rs} \quad \forall r, s \in \{1, \ldots, n\}.$$

Then, by [13] and the lower semicontinuity of the rank function, we obtain

$$\mathscr{B}(\alpha, \beta, P) \cap \overline{\mathscr{B}(\alpha', \beta', P')} \neq \emptyset \Longrightarrow \alpha \prec_K \alpha', \beta \prec_K \beta', P \prec_w P'.$$

Therefore the partial ordering on the set of Bruhat strata

$$(\alpha, \beta, P) \prec (\alpha', \beta', P') :\iff \alpha \prec_K \alpha', \beta \prec_K \beta', P \prec_w P'$$

satisfies the weak adherence property. This, together with the preceding results, implies

Theorem 6.1. *The Bruhat cell decomposition defines a decomposition of* $\mathrm{Hank}_{MN}(n, m, p)$ *into finitely many disjoint cells* $C_i(\alpha, \beta, P)$. *There is a partial ordering* $\prec$ *on the set of Bruhat cells satisfying the weak adherence property*

$$C_i(\alpha, \beta, P) \cap \overline{C_j(\alpha', \beta', P')} \neq \emptyset \Longrightarrow (\alpha, \beta, P) \prec (\alpha', \beta', P').$$

For $m = p = 1$, *the Bruhat cell decomposition coincides with the Fuhrmann–Krishnaprasad cell decomposition.*

The result shows that we have indeed found the proper generalization of the Fuhrmann–Krishnaprasad cell decomposition to the space of matrix-valued transfer functions. Certainly a lot of problems still remain to be investigated. For example, is the Bruhat cell decomposition a cell decomposition in the strict sense, i.e. does it satisfy the strict adherence property

$$C_i \cap \overline{C_j} \neq \emptyset \iff C_i \subset \overline{C_j}$$

and what is the adherence order on the Bruhat cells? Moreover, what is the dimension of a Bruhat cell? Certainly the maximal dimension is $n(m + p)$, but what is the minimal possible dimension of the Bruhat cells? We conjecture that it is equal to $n + m + p - 1$, which is known to be true in the single-input single-output case. We leave these problems open for future research.

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